9.1 Intro to Differential Equations

A **differential equation** is an equation involving derivatives.

A **solution to a differential equation** is any function that satisfies the equation.

Entry Task:
Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer

Example

Consider the differential equation:

$$\frac{dP}{dt} = 2P$$

(a) Is $P(t) = 8e^{2t}$ a solution?

(b) Is $P(t) = t^3$ a solution?

(c) Is $P(t) = 0$ a solution?

The **general solution** to

$$\frac{dP}{dt} = 2P$$

is

$$P(t) = Ce^{2t},$$

for any constant C.

Example: Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

(b) Is $y = t\,e^{-t}$ a solution?

(c) There is a sol'n that looks like

$$y = e^{rt}.$$

Can you find a value of *r* that works?

Application Notes:

$\frac{dy}{dt}$ = “instantaneous **rate of change**

of *y* with respect to *t* ”

“A is proportional to B” means

A = kB, where k is a constant.

In other words, A/B = k.

1. Natural Unrestricted population

Assumption: "*The rate of growth of a population is proportional to the size of the population*."

P(t) = the population at year *t*,

$\frac{dP}{dt} =$ the rate of change of the population with respect to time (i.e. rate of growth).

So the assumption is equivalent to

$$\frac{dP}{dt} = kP,$$

for some constant *k*.

2. Newton's Law of Cooling

Assumption: "*The rate of cooling is proportional to the temperature difference between the object and its surroundings*."

$T_s =$ constant temp. of surroundings

$T(t) =$ temp. of the object at time *t*,

$\frac{dT}{dt} =$ rate of change of temp. with respect to time (*i.e.* cooling rate).

$T - T_s =$ temp. difference between object and surroundings.

Newton's Law of Cooling is equivalent to

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant *k*.

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water.

A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 grams of salt per gal. The vat is thoroughly mixed together.

At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let y(t) = grams of salt in vat at time *t*.

$\frac{y(t)}{50} =$ salt per gallon in vat at time, *t*.

$\frac{dy}{dt} =$ the rate (g/min) at which salt is changing with respect to time.

Thus,

RATE IN = $\left(3\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = 6\,\frac{\text{g}}{\text{min}}$

RATE OUT = $\left(\frac{y}{50}\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = \frac{y}{25}\,\frac{\text{g}}{\text{min}}$

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$

4. All motion problems!

Consider an object of mass *m* kg moving up and down on a straight line.

Let y(t) = `height at time *t*'

$\frac{dy}{dt}$ = `velocity at time *t*'

$\frac{d^2y}{dt^2}$ = `acceleration at time *t*'

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m\frac{d^2y}{dt^2} = \text{sum of forces on the object}$$

Only taking into account gravity we get

$$m\frac{d^2y}{dt^2} = -mg$$

Now consider gravity and ***air resistance***.

One of the most common models is to assume the force due to air resistance is proportional to velocity and in the opposite direction of velocity.

Then we get

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$

5. Many, many others:

Example:

A common assumption for melting snow/ice is "the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area."

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for *r*.